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Measuring Lifetime Poverty

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Abstract: This paper presents an axiomatic framework for measuring lifetime poverty over multiple periods. For an individual, we argue that lifetime poverty is influenced by both the "snapshot" poverty of each period and the poverty level of the "permanent" lifetime consumption; it is also influenced by how poverty spells are distributed over the lifetime. For a society, we consider a path-independence requirement to make the two alternative approaches of aggregation consistent. We axiomatically characterize classes of lifetime poverty indices and derive dominance conditions of poverty orderings for both individual and societal lifetime poverty measurements.

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Measuring Lifetime Poverty

I. Introduction

Suppose two individuals live through the same number of periods of time and both are poor during some of the periods, under what circumstances can we say that one person is poorer than the other from the perspective of lifetime? The current literature on poverty measurement is of little help in answering the question except perhaps in the case where one of them has more income than the other for all periods. In this case, it is reasonable to conclude that the person with more income has less lifetime poverty than the other person. But as long as the two individuals' poverty spells are different (*i.e.*, one person lives in poverty while at the same time the other lives out of poverty) or neither individual has more income for all of the poverty periods, little can be said about lifetime poverty comparison since the comparison requires the evaluation of poverty over time – and there are no axioms in the literature to enable that. To expand the literature of poverty measurement to lifetime poverty, therefore, it is necessary to investigate the notion of lifetime poverty for an individual as well as for a society, and to identify the appropriate axioms for their measurement.

This paper attempts to provide such an axiomatic framework. The measurement for individual lifetime poverty consists of two steps: the measurement of each individual's "snapshot poverty" at each period and the aggregation of these snapshot poverty across all periods. Here a period is interpreted as the basic unit of time that poverty is measured; income is collected at the beginning of each period to enable consumption in that period and in the subsequent periods. A person is poor in a period if and only if his consumption level in that period falls short of the poverty line. For the measurement of lifetime poverty, it is important to stress the use of consumption rather than the disposable income in the calculation of poverty as consumption is a much more accurate measure of the standard of living over time. The measurement of the snapshot poverty at each period is straightforward; each individual's poverty is measured as his consumption deprivation from the poverty line. The current literature of poverty measurement provides ample guidelines for this stage of the measurement.

It is the second stage of the lifetime poverty measurement that expands the literature of poverty measurement. When viewed from the perspective of lifetime, the suffering and deprivation of each individual in each period transmits into the lifetime evaluation of poverty. All other things equal, the more deprivation a person endures in a given period, the more lifetime poverty it will be for the individual. This "experience axiom" is akin to the monotonicity axiom or the subgroup consistency axiom used in the measurement of snapshot poverty. All these snapshot poverty experiences, however, may not suffice to determine lifetime poverty since an individual may live out of poverty for some periods of the lifetime. Given that the deprivation in one period can be offset by the experience of affluent living in another period, lifetime poverty is also influenced by the consumption of lifetime as a whole when it is compared with a sort of "lifetime poverty line." The essence of this argument is reflected in the "lifetime axiom" which stipulates that lifetime poverty is a function of the lifetime permanent consumption poverty (in the paper the permanent consumption is approximated by the average lifetime consumption in the absence of discounting). With other regularity assumptions, we derive a general class of lifetime poverty indices that are the weighted sum between the average snapshot poverty level across all periods and the poverty of average lifetime consumption.

To further characterize the class of lifetime poverty indices, we propose two additional axioms governing the aggregation of snapshot poverty across time. The first of these we call the early poverty axiom. There is substantial evidence that poverty in earlier stages of life affects consumption in later periods and leaves a deeper mark on lifetime deprivation. Therefore, we require that the weight attached to each period's poverty deprivation be a nonincreasing function of consumption of the time period in which the individual lives. Our reason for putting forward this "early poverty axiom" is to reflect the concern that poverty earlier in life has a greater life-long impact on an individual well-being for a combination of reasons including its negative impact on both physical and psychological health of the individual. The fact that early poverty worsens an individual's capability to generate higher consumption later in life is captured implicitly by the fact that we include all future levels of consumption in our measurement of lifetime poverty. Thus, for example, there are two channels through which one can argue that alleviation of childhood poverty should receive particular attention: Firstly, due to the relative importance per se of deprivation suffered earlier rather than later in life. Secondly, to enhance future productivity and consumption by individuals. Our focus is to develop a measurement approach that accounts for this first reason. The second reason would be captured through the way that lifetime productivity or incomes of individuals evolve and should be modelled in the context of whatever poverty reduction policy is being considered. Note that if, alternatively, one feels that poverty at both early and late life should be weighted more - say due to the special vulnerability of the young and old to spells of poverty, then the weighting function should be U-shaped. It would, of course, be possible to adopt such an axiom and analyze its impact in much the same way as we analyze the impact of our early poverty axiom.

Another common concern with poverty when considered in a lifetime context is that of chronic poverty; that is, frequent poverty spells separated by few periods of nonpoverty. We reflect this concern by formalizing a "chronic-poverty axiom." This axiom states that an individual suffers more from living in poverty for consecutive periods than living in and out of poverty alternatively. The axiom implies that the weighting function is concave in the time period. When all members of the class are considered for poverty orderings, we derive the corresponding dominance conditions that are related to the familiar "concentration dominances."

Conceptually, the measurement of lifetime poverty for a society can have two alternative approaches to follow: (1) measuring each individual's lifetime poverty first and then aggregating across all individuals in the society; and (2) measuring society's aggregate poverty in each period and then aggregating them across time in a manner similar to the measurement of individual lifetime poverty. These two approaches represent two different paths to obtain the same societal lifetime poverty index. It is useful to consider the situation where the two approaches yield the same result. This path-independence consideration leads to a specific functional form for the societal lifetime poverty index. When all path-independent lifetime poverty indices are considered, a set of poverty ordering conditions are also derived. In these conditions, income mobility – income movement among people over time – is shown to play a role in determining a society's lifetime poverty; all other things equal, a more mobile society tends to have less aggregate lifetime poverty.

The topic of the paper is related to that of Rodgers and Rodgers (1993) where their focus is on the measurement of chronic poverty. To measure the aggregate poverty of the society, they use the weighted average snapshot poverty of the society across the time periods considered; the weight is the proportion of population presented in each time period. As we have explained above, this simple averaging process is far from adequate in measuring the lifetime poverty of a society. Our paper is also related to the two papers by Karcher, Moyes and Trannoy (1995, 2002) where they measure society's aggregate social welfare over time with different discounting concerns. Unlike their characterizations, our approach does not reply on discounting concern and is axiomatically characterized. Also income/consumption mobility plays a role in our measurement/rankings of lifetime poverty.

The rest of the paper is organized as follows. The next section characterizes individual lifetime poverty. Here we propose a set of axioms that are pertinent to measuring poverty over time. Section III considers the aggregation of individual lifetime poverty across the society. Section IV provides some additional remarks and also concludes the paper.

II. The Measurement of Individual Lifetime Poverty

Consider an individual who lives through T periods. Each period can be interpreted as a year or as a phase of life such as youth, middle age and old age. In each period t, $t = 1, 2, ..., T$, the individual has a non-negative level of consumption x_t. In each period, the individual's poverty status is determined by comparing his consumption level with the poverty line $0 < z < \infty$ which is exogenously given and remains constant throughout the T periods. The individual is poor in period t if and only if his consumption level x_t is strictly less than z. Denote $\mathbf{x} = (x_1, x_2, ..., x_T)'$ the profile of the individual's lifetime consumptions, his lifetime poverty level is a function $P(\mathbf{x}; z)$ which maps each consumption profile $\mathbf{x}$ into $[0, \infty)$. The average consumption of the individual over the T periods is $\bar{x}$. For each consumption variable, we also define its censored consumption as $\tilde{x}_t = \min\{x_t, z\}$. In what follows, we discuss the appropriate axioms that can be imposed upon the functional form of $P(\mathbf{x}; z)$.

At the beginning of each period, the individual collects income and allocates it to the consumptions of that period and the periods to come; at the end of each period, the individual compares his consumption level x_t with the poverty line z. If $x_t < z$, he has poverty deprivation which is measured by $p(x_t; z)$: $p(x_t; z) > 0$ if $x_t < z$; otherwise he lives out of poverty: $p(x_t; z) = 0$ if $x_t \geq z$. We refer to $p(x_t; z)$ as the "snapshot poverty" of period t. The measurement of poverty deprivation has been well studied in the literature. In general, we assume that $p(x_t; z)$ is continuous, $\frac{\partial p(x_t;z)}{\partial x_t} < 0$ and $\frac{\partial^2 p(x_t;z)}{\partial x_t^2} > 0$ for all $x_t \in [0, z)$.[1] That is, poverty deprivation decreases as consumption increases; it decreases, however, at a slower pace as consumption increases – in part this is to reflect the poverty aversion consideration (Zheng, 2000). Although higher-order conditions can be entertained, in this paper we limit our investigation to only the first two orders.[2] Accordingly, we also assume the lifetime poverty measure $P(\mathbf{x}; z)$ to exhibit similar properties as $p(x; z)$, *i.e.*, $\frac{\partial P(\mathbf{x};z)}{\partial x_t} \leq 0$ and $\frac{\partial^2 P(\mathbf{x};z)}{\partial x_t^2} \geq 0$ for all $x_t \in [0, \infty)$. Note that here we require only weak inequalities and the range for x_t is over $[0, \infty)$ rather than $[0, z)$. This is because, unlike in the measurement of snapshot poverty where any change in x_t above z has no effect on the poverty level, here such a change may or may not affect the lifetime poverty – as we will see below.

Our first axiom establishes the connection between the "snapshot poverty" of each period $p(x_t; z)$ and that of lifetime $P(\mathbf{x}; z)$.

The experience axiom: $P(\mathbf{x}; z)$ is an increasing function of $p(x_t; z)$ for $t = 1, 2, ..., T$.

What this axiom states is pretty clear: the suffering from poverty deprivation in each period is transmitted directly and positively into the lifetime poverty deprivation. Recounting at the end of the Tth period, the individual may feel life has been harder if he had a poor childhood than otherwise – all else the same. This is certainly a reasonable requirement: if $p(x_t; z)$ increases then it must be $x_t < z$ and x_t decreases which, in turn, must increase the lifetime poverty deprivation. This interpretation is the usual monotonicity axiom used in the snapshot poverty measurement applied to the lifetime context. This axiom is also akin to the familiar subgroup consistency axiom which says that if a subgroup experiences an increase in poverty, then the overall society's poverty ought to go up. Viewing the individual's consumption in each year as a subgroup and the lifetime consumption profile as a society, then the experience axiom becomes a direct implication of the subgroup consistency axiom.

But the individual's picture of lifelong living is not entirely dictated by the poverty

[1] Other axioms in the literature such as the increasing poverty line axiom – $p(x_t; z)$ is increasing in z – and the unit-consistency axiom – which implies that $p(x_t; z)$ is a homogeneous function of x_t and z - may also be considered to further specify the functional form of $p(x_t; z)$.

[2] Our choice also reflects the fact that poverty orderings at third and above orders may collapse to second-order if the poverty line is uncertain and expands over a large interval (Zheng, 1999). With uncertain poverty lines, the consideration of higher-than-second-order conditions may introduce little additional insights on poverty orderings.

spells that he has experienced in the various periods. He might view that "even though I had a rough childhood, life as a whole has been very good to me since I had an affluent living later in my life." This means that the individual registers all poverty deprivations but he also allows rich livings in the rest of his lifetime to offset some of the bad memories. How to capture this offsetting phenomenon? A natural and manageable way is to consider poverty when the entire lifetime is considered as a whole and the consumption over the lifetime can be completely smoothed out. The poverty level is then computed by comparing the lifetime (permanent) consumption with the lifetime (permanent) poverty line. Since we assume the poverty line remains the same throughout all T periods, all consumption levels will not be discounted either (or consider they are already discounted). It follows that we can proxy the permanent consumption with a simple average consumption over the lifetime.[3] Noting that $p(\bar{x}; z)$ is the poverty of lifetime average consumption, our second axiom summaries the afore-discussed influence of lifetime smoothed consumption on the lifetime poverty.

The lifetime axiom: $P(\mathbf{x}; z)$ is an increasing function of $p(\bar{x}; z)$.

With the lifetime axiom, the focus axiom in the literature – any change made in any above-the-poverty-line consumption has no affect on the poverty level – needs to be modified. Since now a change in a consumption may affect lifetime poverty through two routes: through the snapshot poverty in each period and through the poverty of average-lifetime-consumption. It follows that the new focus axiom should be reformulated as: a change in a period's consumption has no effect on lifetime poverty if and only if both the consumption level and the average-consumption level of the entire lifetime are above the poverty line. This is to say $\frac{\partial P(\mathbf{x};z)}{\partial x_t} < 0$ if either $x_t < z$ or $\bar{x} < z$; otherwise $\frac{\partial P(\mathbf{x};z)}{\partial x_t} = 0$ – as we have assumed at the beginning of this section.

The experience axiom and the lifetime axiom together imply that the lifetime poverty $P(\mathbf{x}; z)$ is an increasing function of $p(x_t; z)$ for $t = 1, 2, .., T$ and $p(\bar{x}; z)$. To fully calibrate the relation among the three types of poverty, a stronger axiom than both the experience axiom and the lifetime axiom is needed.

The strong monotonicity axiom: for two consumption profiles $\mathbf{x}$ and $\mathbf{y}$, $P(\mathbf{x}; z) > P(\mathbf{y}; z)$ if $p(x_t; z) \geq p(y_t; z)$ for $t = 1, 2, .., T$ and $p(\bar{x}; z) \geq p(\bar{y}; z)$ with at least one inequality holding strictly.

The strong monotonicity axiom implies that the lifetime poverty is uniquely determined by $\{p(x_t; z)\}$ and $p(\bar{x}; z)$. Clearly, the strong monotonicity axiom implies both of the previous axioms but it is not implied by either one of them or both of

[3] If a more suitable representation of permanent consumption is deemed necessary, we can replace $\bar{x}$ with such a permanent-consumption function $\mu(x_1, x_2, ...x_T)$ in the rest of the paper. All results involving $\bar{x}$ will also hold with some appropriate modifications. For example, one condition at the end of the next section – $F(x_1, ..., x_t, ...x_T) \geq G(x_1, ..., x_t, ...x_T)$ for all $x_1, ..., x_t, ...x_T$ such that $\bar{x} < z$ – becomes $F(x_1, ..., x_t, ...x_T) \geq G(x_1, ..., x_t, ...x_T)$ for all $x_1, ..., x_t, ...x_T$ such that $\mu(x_1, x_2, ...x_T) < z$.

them. To see how the conjunction of snapshot poverty and lifetime poverty operate differently from consideration of one on its own, consider the following example.

Example: Let $z = 5$ and $T = 3$. We have for $\mathbf{x} = (1, 3, 7)$ and $\mathbf{y} = (2, 4, 8)$ both that $p(x_t; z) \geq p(y_t; z)$ for all t (with strict inequality for some t), and $p(\bar{x}; z) \geq p(\overline{y}; z)$ since $\overline{x} \leq \overline{y}$. Moreover, in this case we have $\overline{x} < \overline{y} < z$ which implies $p(\bar{x}; z) > p(\overline{y}; z)$. Thus, even though we don't have $x_t \leq y_t$, $\forall t$, the evaluation of snapshot poverty and lifetime poverty operate in the same direction and we have $P(\mathbf{x}; z) > P(\mathbf{y}; z)$ according to the strong monotonicity axiom in conjunction with the lifetime and experience axioms. However, if we consider $\mathbf{x}' = (1, 3, 14)$ we have that in considering snapshot poverty $\mathbf{x}'$ displays more poverty than $\mathbf{y}$ but the opposite applies in comparing lifetime poverty. Thus, in this case the evaluation of snapshot and lifetime poverty do not agree and we cannot rank these two distributions.

But in what way may these terms jointly determine the lifetime poverty? For each period's snapshot poverty $p(x_t; z)$, we assume that its effect on the lifetime poverty $P(\mathbf{x}; z)$ is independent of any other period's snapshot poverty $p(x_s; z)$ for $s \neq t$ and of the poverty level of the average lifetime consumption $p(\bar{x}; z)$. This axiom is akin to the decomposability axiom used in the snapshot poverty measurement; it enables researchers to compute the contribution of each year's consumption and the smoothed consumption to total poverty and allows policy-makers to identify the specific factors that are responsible for changes in the overall poverty value. This requirement amounts to saying that there is no interaction among the poverty levels of the various periods and that of the average consumption. Formally, this assumption can be stated as follows.

The independence axiom: $\frac{\partial^2 P(\mathbf{x};z)}{\partial p(x_t;z) \partial p(x_s;z)} = 0$ for all $s \neq t$ and $\frac{\partial^2 P(\mathbf{x};z)}{\partial p(x_t;z) \partial p(\bar{x};z)} = 0$ for all $t = 1, 2, ..., T$.

Example: To illustrate the implications of the independence axiom, consider the following consumption vectors; $\mathbf{x} = (1, 1, 4, 12)$, $\mathbf{y} = (2, 1, 2, 13)$ and $\mathbf{x}' = (1, 1 + \delta, 4, 12)$, $\mathbf{y}' = (2, 1 + \delta, 2, 13)$, $\forall \delta \in [-1, M)$ for any finite $M > 0$. The independence axiom implies that if $P(\mathbf{x}; z) = P(\mathbf{y}; z)$ then $P(\mathbf{x}'; z) = P(\mathbf{y}'; z)$. The reason is that altering income in period $t = 2$, which is equal for the two distributions, in the same way has the same effect on snapshot poverty. Moreover, $\overline{x} = \overline{y}$ and so the change in income applied to obtain $\mathbf{x}'$ and $\mathbf{y}'$ also has the same effect on lifetime poverty. The independence axiom means that these are the only two implications of this change in consumption on $P(\mathbf{x}; z)$ and $P(\mathbf{y}; z)$ and so the resulting poverty levels for $\mathbf{x}'$ and $\mathbf{y}'$ must also be equal. Without the independence axiom, these (equivalent) changes in $p(x_2; z)$ and $p(y_2; z)$ could have changed snapshot poverty in any of the other periods $t = 1, 3$ differently for the two distributions and this would imply possibly different poverty levels for distributions $\mathbf{x}'$ and $\mathbf{y}'$. A similar argument could apply if the means of lifetime consumptions were different.

If the individual's consumption level is equal in all periods, *i.e.*, $x_s = x_t$ for all s and t, then it is reasonable to say that this individual's lifetime poverty can be appropriately represented by the snapshot poverty $p(x_t; z)$. This axiom is similar to the normalization axiom used in the measurement of income inequality.

The normalization axiom: $P(\mathbf{x};z) = p(x;z)$ if $x_1 = x_2 = ... = x_T$.

The following proposition documents the implication of the above three axioms for the functional form of $P(\mathbf{x};z)$.

Proposition 2.1. A lifetime poverty measure $P(\mathbf{x};z)$ satisfies the strong monotonicity axiom, the independence axiom and the normalization axiom if and only if there exist continuous and positive functions $f_1, f_2, ..., f_T$ and g such that

$$P(\mathbf{x};z) = \sum_{t=1}^{T} f_t(t)p(x_t;z) + g(T)p(\bar{x};z) \tag{2.1}$$

where $\sum_{t=1}^{T} f_t(t) + g(T) = 1$.

Proof. The strong monotonicity axiom implies that $P(\mathbf{x};z)$ is an increasing function of only $\{p(x_t;z)\}$ and $p(\bar{x};z)$. That is, there exists a continuous and increasing function f such that

$$P(\mathbf{x};z) = f[p(x_1;z), p(x_2;z), ..., p(x_T;z), p(\bar{x};z)]. \tag{2.2}$$

The independence axiom implies that $P(\mathbf{x};z)$ is additively separable in $\{p(x_t;z)\}$ and $p(\bar{x};z)$, *i.e.*,

$$P(\mathbf{x};z) = \tilde{f}_1[p(x_1;z)] + \tilde{f}_2[p(x_2;z)] + ...\tilde{f}_T[p(x_T;z)] + \tilde{g}_T[p(\bar{x};z)] \tag{2.3}$$

for some continuous and positive functions $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_T$, and $\tilde{g}_T$.

In the case where $x_1 = x_2 = ... = x_T$, denote $\xi = p(x_t;z) = p(\bar{x};z)$, then the normalization axiom entails

$$\sum_{t=1}^{T} \tilde{f}_t(\xi) + \tilde{g}_T(\xi) = \xi \tag{2.4}$$

for all $\xi \in (0, p(0,z)]$. Differentiating equation (2.4) twice with respect to ξ yields

$$\sum_{t=1}^{T} \tilde{f}_t^{''}(\xi) + \tilde{g}^{''}(\xi) = 0. \tag{2.5}$$

By choosing a $\mathbf{x}$ such that $x_t < z$ and $x_s \geq z$ for all $s \neq t$ with $\bar{x} \geq z$, equation (2.3) becomes

$$P(\mathbf{x};z) = \tilde{f}_t[p(x_t;z)]. \tag{2.3a}$$

The assumption $\frac{\partial^2 P(\mathbf{x};z)}{\partial x_t^2} \geq 0$ implies

$$\tilde{f}_t^{''}(p')^2 + \tilde{f}_t^{'} p^{''} \geq 0.$$

Since the functional forms of $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_T$, and $\tilde{g}_T$ are independent of the deprivation function p used, by choosing a deprivation function p such that p'' can be arbitrarily close to zero (say $p(x_t; z) = z - x_t$ or better the index given in Zheng (1999, Equation (A3), p. 369)), we have

$$\tilde{f}_t'' \geq 0 \text{ for all } t = 1, 2, ..., T.$$

Next, choosing a $\mathbf{x}$ with $x_1 > z$ but $\bar{x} < z$, we have

$$P(\mathbf{x}; z) = \tilde{f}_1[0] + \tilde{f}_2[p(x_2; z)] + ...\tilde{f}_T[p(x_T; z)] + \tilde{g}[p(\bar{x}; z)]. \tag{2.3b}$$

Again, the requirement of $\frac{\partial^2 P(\mathbf{x};z)}{\partial x_1^2} \geq 0$ implies

$$[\tilde{g}''(p')^2 + \tilde{g}'p'']/n^2 \geq 0,$$

and once again it must be the case

$$\tilde{g}'' \geq 0.$$

This means all items in (2.5) cannot be negative. It follows that every one of them must be zero, or

$$\tilde{f}_t(\xi) = f_t(t)\xi \text{ for all } t = 1, 2, ..., T \text{ and } \tilde{g}_T(\xi) = g(T)\xi \tag{2.6}$$

for some continuous and positive functions $f_t(t)$, $t = 1, 2, ..., T$, and $g(T)$. Substituting $\tilde{f}_t()$ and $\tilde{g}()$ back into (2.3) completes the proof of the proposition. □

Proposition 1 reveals that if a lifetime poverty index satisfies the three axioms proposed above, the marginal contributions from each period's poverty as well as from the poverty of average-lifetime-consumption are independent of the contributors. Since $\sum_{t=1}^{T} f_t(t) + g(T) = 1$ by the normalization axiom and each coefficient is non-negative and $g(T) > 0$, we can further express (2.1) as

$$P(\mathbf{x}; z) = [1 - g(T)] \left\{ \sum_{t=1}^{T} \frac{f_t(t)}{1 - g(T)} p(x_t; z) \right\} + g(T)p(\bar{x}; z). \tag{2.1a}$$

Denote $\alpha(t, T) = \frac{f_t(t)}{1-g(T)}$ and $\beta(T) = 1 - g(T)$, (2.1a) states that $P(\mathbf{x}; z)$ is a weighted average between $\sum_{t=1}^{T} \alpha(t, T)p(x_t; z)$ and $p(\bar{x}; z)$ with the weights being $\beta(T)$ and $1 - \beta(T)$, respectively. Note that $\sum_{t=1}^{T} \alpha(t, T)p(x_t; z)$ is a weighted average of all snapshot-poverty levels of the T periods with $\sum_{t=1}^{T} \alpha(t, T) = 1$. This observation leads to the following corollary.

Corollary 2.1. A lifetime poverty measure $P(\mathbf{x}; z)$ satisfies the strong monotonicity axiom, the independence axiom and the normalization axiom if and only if it can be written as a weighted average between the weighted average of all snapshot poverty levels and the poverty of average lifetime consumption

$$P(\mathbf{x}; z) = \beta(T) \left\{ \sum_{t=1}^{T} \alpha(t, T)p(x_t; z) \right\} + [1 - \beta(T)]p(\bar{x}; z). \tag{2.1b}$$

for some $\alpha(t,T)$ and $\beta(T)$ such that $0 < \alpha(t,T) < 1$, $0 < \beta(T) < 1$ and $\sum_{t=1}^{T} \alpha(t,T) = 1$.

Examples of lifetime poverty indices: $\alpha(t,T) = (1 - \frac{t}{T+1})^{\gamma}$ and ρ^t with $0 < \rho < 1$; $p(x_t; z) = (1 - x_t/z)^{\varepsilon}$, $1 - (x_t/z)^{\varepsilon}$ and $\ln(z - x_t + 1)$ for $x_t < z$.

The coefficient $\beta(T)$ plays the role of balancing between the average "snapshot poverty" and the average-lifetime-consumption poverty: a larger value of $\beta(T)$ means that the individual concerns more about the poverty incidences he has experienced and less about when the life as a whole is evaluated. In the limiting case of $\beta(T) = 1$, the individual gives no consideration to the average or smoothed lifetime consumption; the lifetime poverty is determined exclusively by the poverty deprivations he has had in his life no matter how affluent he may be when life as a whole is judged. On the other hand, $\beta(T) = 0$ means that the individual cares about only the lifetime aggregate or average consumption and the poverty deprivation in any period matters nothing in his evaluation of lifetime poverty. In this sense, we may label $\beta(T)$ as a "memory parameter" – $\beta(T) = 1$ is the polar case of "perfect memory" and $\beta(T) = 0$ is the other polar case of "perfect no memory," respectively. To compute the individual's lifetime poverty index using (2.1b) – which will be used in the rest of the paper, the memory parameter must be specified. If all possible values of $\beta(T)$ are considered, then we have

Corollary 2.2. For two lifetime consumption profiles $\mathbf{x}$ and $\mathbf{y}$, $P(\mathbf{x}; z) \geq P(\mathbf{y}; z)$ for any poverty measure of the form (2.1b) and for all $\beta(T) \in (0,1)$ if and only if

$$\sum_{t=1}^{T} \alpha(t,T) p(x_t; z) \geq \sum_{t=1}^{T} \alpha(t,T) p(y_t; z) \text{ and } p(\bar{x}; z) \geq p(\bar{y}; z). \tag{2.7}$$

This result can be regarded as our first dominance condition. The implication of the corollary is reasonable and intuitive. To characterize further the lifetime poverty index and establish additional dominance conditions, we need to take a closer look at the weighting function $\alpha(t,T)$. It is this task that we turn our attention to now.

For a given lifetime consumption profile, suppose the individual can choose a permutation of the T consumption levels so as to minimize the lifetime poverty $P(\mathbf{x}; z)$, what would the individual choose? Clearly, any permutation will not affect the poverty of average-lifetime-consumption since the average consumption remains the same. It is the average "snapshot" poverty that can be affected and the values of the coefficients $\alpha(t,T)$ dictate the chosen permutation. If $\alpha(t,T)$s are equal, *i.e.*, $\alpha(s,T) = \alpha(t,T)$ for all s and t, then permutation of the consumption profile again does matter; otherwise, higher consumptions should be allocated to the periods with greater α-values.

In general, higher values of $\alpha(t,T)$ reflect the periods that are more critical for an individual's lifetime well-being. If the individual (or ethical observer) cannot sort out the order of importance of all the periods, then there is no agreeable ranking for the α-values. When each $\alpha(t,T)$ can assume any nonnegative value, we have the following result on the poverty orderings by all possible lifetime poverty measures (2.1b). The proof of the proposition is straightforward and is thus not furnished.

Proposition 2.2. For two lifetime consumption profiles $\mathbf{x}$ and $\mathbf{y}$, $P(\mathbf{x}; z) \geq P(\mathbf{y}; z)$ for all $\beta(T) \in [0, 1]$ and all possible values of $\alpha(t, T)$s if and only if

$$\tilde{x}_t \leq \tilde{y}_t \text{ and } \min\{\bar{x}, z\} \leq \min\{\bar{y}, z\} \tag{2.8}$$

for $t = 1, 2, ..., T$ and the strict inequality holds at least once. Note that $\tilde{x}_t = \min\{x_t, z\}$.

Condition (2.8) states that for profile $\mathbf{x}$ to have more lifetime poverty than $\mathbf{y}$ unambiguously (*i.e.*, for all possible weights), it must be the case that the consumption in each period of $\mathbf{x}$, if it is below the poverty line, is no greater than that in $\mathbf{y}$; moreover, it has also to be the case that the average lifetime consumption, if it is below the poverty line, is no greater in $\mathbf{x}$ than that in $\mathbf{y}$. Note that the first set of conditions does not necessarily imply the last condition because consumptions are censored at the poverty line.

Examples: For distributions of consumption $\mathbf{x} = (1, 2, 6)$ and $\mathbf{y} = (2, 2, 3)$ with $z = 3$ note that the first set of conditions ($\tilde{x}_t \leq \tilde{y}_t$) of Proposition 2.2 are satisfied, but not the second. Thus, for $\beta(T)$ small enough (i.e. $1 - \beta(T)$ large enough), it is possible that $P(\mathbf{x}; z) \leq P(\mathbf{y}; z)$. To illustrate the importance of the condition that the poverty comparison is satisfied for all possible values of $\alpha(t, T)$, consider the following example with $z = 5$ and $T = 3$, with $\mathbf{x} = (4, 3, 1)$ and $\mathbf{y} = (1, 4, 4)$. Note that for equal weights $\alpha(t, T)$ we have $P(\mathbf{x}; z) > P(\mathbf{y}; z)$ since $p(x_1; z) = p(y_2; z)$, $p(x_2; z) > p(y_3; z)$ and $p(x_3; z) = p(y_1; z)$. However, the greatest degree of poverty in the $\mathbf{x}$ distribution occurs earlier in time than for the $\mathbf{y}$ distribution. Applying the weights, for example, of ρ^t for $\rho = 0.8$ and using the poverty index $p(x; z) = \max\{z - x, 0)$ we obtain $P(x; z) < P(y; z)$. Thus, we cannot order $\mathbf{x}$ and $\mathbf{y}$ unambiguously (i.e., for all values of weights $\alpha(t, T)$.

If the individual concludes that all periods are equally important to him, *i.e.*, $\alpha(s, T) = \alpha(t, T)$ for all s and t, then the following proposition can be easily verified using standard results from the literature of majorization.

Proposition 2.3. For two lifetime consumption profiles $\mathbf{x}$ and $\mathbf{y}$, if $\alpha(t, T)$s are the same, then $P(\mathbf{x}; z) \geq P(\mathbf{y}; z)$ for all $\beta(T) \in [0, 1]$ if and only if

(1) $\sum_{t=1}^{T} p(x_t, z) \geq \sum_{t=1}^{T} p(y_t, z)$ and $\min\{\bar{x}, z\} \leq \min\{\bar{y}, z\}$ for a given deprivation function p; or

(2) also for all deprivation functions p such that $p' < 0$, vector $(\tilde{x}_1, ..., \tilde{x}_T)$ is rank dominated by vector $(\tilde{y}_1, ..., \tilde{y}_T)$ and $\min\{\bar{x}, z\} \leq \min\{\bar{y}, z\}$; or

(3) also for all deprivation functions p such that $p' < 0$ and $p'' > 0$, vector $(\tilde{x}_1, ..., \tilde{x}_T)$ is generalized Lorenz dominated by vector $(\tilde{y}_1, ..., \tilde{y}_T)$ and $\min\{\bar{x}, z\} \leq \min\{\bar{y}, z\}$.

If the individual (or ethical observer) can successfully rank order the importance of his well-being in all periods with no two periods equally important,[4] then condition (2.8) can be refined. It seems that there is a consensus that early stages of life such as childhood matter more than later-life periods in shaping the individual's lifetime well-being/poverty. It can be argued that more poverty a person suffers in his childhood or less education he receives in his youth, the less is the person's future well-being conditional on any given level of consumption. There are several reasons for such an effect. For example, poorer health or poorer psychological well-being including the effect that arrises from less education. Of course, there are functional implications of early poverty on future consumption due to reduced capabilities. As noted earlier, this impact is taken into account in our measurement of lifetime consumption directly through the inclusion of the entire lifetime consumption vector in $P(\mathbf{x}; z)$. Naturally, consumption and poverty later in life cannot impact well-being earlier in life. Translated in terms of the weighting function $\alpha(t, T)$, this notion can be formally stated as an axiom.

The early-poverty axiom: the weighting function $\alpha(t, T)$ is nonincreasing in time t.

Note that what the early-poverty axiom states is different from the discounting concern that is usually imposed on aggregation over time, although the discounting weight $\alpha(t, T) = \rho^t$ with $0 < \rho < 1$ does satisfy the axiom. Here we do not discount over time; our concern here is purely about the size of impact of each period's poverty deprivation on the aggregate lifetime poverty. In fact, as we will see later, the usual discount-weighting scheme is ruled out when a further axiom is introduced. As for the other type of weighting function mentioned earlier, *i.e.*, $\alpha(t, T) = (1 - \frac{t}{T+1})^\gamma$, $\gamma > 0$ is required for it to satisfy the early-poverty axiom.[5] With this additional axiom, we have

Proposition 2.4. For two lifetime consumption profiles $\mathbf{x}$ and $\mathbf{y}$, $P(\mathbf{x}; z) \geq P(\mathbf{y}; z)$ for all $\beta(T) \in [0, 1]$ and all possible values of $\alpha(t, T)$s satisfying the early-poverty axiom if and only if

(1) $\sum_{t=1}^{l} p(x_t, z) \geq \sum_{t=1}^{l} p(y_t, z)$ for $l = 1, 2, ..., T$ and $\min\{\bar{x}, z\} \leq \min\{\bar{y}, z\}$ for a given deprivation function p; or

(2) also for all deprivation functions p such that $p' < 0$, vector $(\tilde{x}_1, ..., \tilde{x}_l)$ is rank dominated by vector $(\tilde{y}_1, ..., \tilde{y}_l)$ for $l = 1, 2, ..., T$ and $\min\{\bar{x}, z\} \leq \min\{\bar{y}, z\}$; or

(3) also for all deprivation functions p such that $p' < 0$ and $p'' > 0$, vector $(\tilde{x}_1, ..., \tilde{x}_l)$ is generalized Lorenz dominated by vector $(\tilde{y}_1, ..., \tilde{y}_l)$ for $l = 1, 2, ..., T$ and $\min\{\bar{x}, z\} \leq \min\{\bar{y}, z\}$.

Proof. The early-poverty axiom is equivalent to requiring $\alpha(1, T) \geq \alpha(2, T) ... \geq$

[4] If any two or more periods are equally important, *i.e.*, their coefficients are the same, then these periods can be grouped together to form a single period and the results derived in Proposition 2.3 and onward apply to this "grouped" consumption profile.

[5] Any given set of weights $\alpha(t, T)$ can be normalized by simply dividing by the sum in order to ensure $\sum_{t=1}^{T} \alpha(t, T) = 1$. Also, a positive constant could be added to $\alpha(t, T)$ for the above example in order to ensure $\alpha(t, T) > 0$ for $t = T$

$\alpha(T,T)$. With this condition, the proof of part (1) follows directly from Abel's partial summation formula (see Rudin (1976, p. 70)). Parts (2) and (3) result from applying the standard rank order condition (Saposnik, 1986) and majorization (Marshall and Olkin, 1979). □

A necessary condition for parts (2) and (3) in the above proposition is

$$\sum_{t=1}^{l} \tilde{x}_t \leq \sum_{t=1}^{l} \tilde{y}_t \text{ for } l = 1, 2, ..., T$$

which is the concentration curve dominance between (censored) lifetime consumption profiles of $\mathbf{x}$ and $\mathbf{y}$. Concentration curve dominance is constructed similarly to the generalized Lorenz curve dominance with the exception that the values of $\{x_t\}$ and $\{y_t\}$ are not sorted before the construction. The concentration curve dominance condition can be handily used to screen out consumption profiles in (lifetime) poverty orderings.

Finally, we introduce the axiom that characterizes the chronic aspect of lifetime poverty. Suppose an individual has to endure two poverty spells around certain periods of time. Would his lifetime poverty be greater if he had to live in poverty consecutively for two periods or alternatively with a period of affluence in between? Chronic poverty is defined as living in poverty continuously for an extended period of time. The issue of chronic poverty has been at the heart of recent poverty research and anti-poverty policy debate (see, for example, Hulme and McKay, 2005; the Chronic Research Center Report, 2005). It is argued that transitory poverty is not as much a cause for concern as chronic poverty. Chronic poverty should be the target for poverty reduction effort because it gives rise to a series of social-economic problems and poses the greater threat to economic growth in the developing countries. Following this notion, we argue that living in poverty for consecutive periods leads to higher lifetime poverty than living in poverty and prosperity alternatively even though the total number of poverty spells and depth of poverty remain the same. This consideration is formally presented in the axiom below.

The chronic-poverty axiom: for two given consumptions $a < z$ and $b < z$, the closer the two spells together, the greater is the resulting lifetime poverty, *i.e.*,

$$\alpha(s,T)p(a;z) + \alpha(u,T)p(b;z) \geq \alpha(r,T)p(a;z) + \alpha(v,T)p(b;z) \tag{2.9}$$

for all $1 \leq r < s \leq u < v \leq T$ such that $s - r = v - u$.

Notice that the chronic-poverty axiom specifies that two spells of poverty occurring in periods (s,u) have greater impact than if the same spells had occurred in periods spread out symmetrically by $k > 0$ periods in both directions; *i.e.*, in periods $(r,u) = (s-k, u+k)$. To understand the reason for adopting this symmetry requirement, consider the implication of not doing so and $s - r \neq v - u$. Specifically, we consider the case of $r = 1$, $s = u = T-1$, and $v = T$ for $T > 3$. Set $a = b$ and note that the chronic-poverty axiom without the normalization would imply

$$\alpha(T-1,T)p(a;z)+\alpha(T-1,T)p(a;z)\geq\alpha(1,T)p(a;z)+\alpha(T,T)p(a;z)$$

Dividing by $p(a;z)$ and re-arranging gives the inequality

$$\alpha(1,T)-\alpha(T,T)\leq 2\left[\alpha(T-1,T)-\alpha(T,T)\right]$$

Therefore, without the requirement $s-r=v-u$ the early poverty axiom would be compromised to the extent that the range of the (nonincreasing) weights over the entire time period $T=1$ to $T=T$ would be restricted to only twice the difference between the weights assigned to the last two time periods. Thus, if the weights for the last two time periods were equal then the "chronic-poverty axiom" without requiring $s-r=v-u$ would imply that the early poverty axiom would be made impotent (*i.e.*, the weights $\alpha(t,T)$ would have to be equal for all t).

The chronic-poverty axiom (*with* the requirement of $s-r=v-u$) implies that the weighting function $\alpha(t,T)$ is concave in t.[6] To see this take the case of $a=b$ in (2.9) to obtain

$$\alpha(s,T)p(a;z)+\alpha(u,T)p(a;z)\geq\alpha(r,T)p(a;z)+\alpha(v,T)p(a;z)$$

Dividing by $p(a;z)$ and rearranging gives

$$\alpha(s-k,T)-\alpha(s,T)\leq\alpha(u,T)-\alpha(u+k,T),\ u\geq s \tag{2.9a}$$

which is satisfied if and only if $\alpha(t,T)$ is concave in t.

To fully adopt the chronic-poverty axiom for poverty measurement, we need to extend the weighting function to period $T+1$: $\alpha(T+1,T)=0$. This is the sort of assumption of "poverty irrelevance of life after death." By letting $a=b$ and $s=u=2,...,T$ and $k=1$ (2.9a), we have

$$0<\alpha(1,T)-\alpha(2,T)\leq...\leq\alpha(T-1,T)-\alpha(T,T)\leq\alpha(T,T).$$

Examples of satisfactory poverty indices: $\alpha(t,T)=(1-\frac{t}{T+1})^{\gamma}$ with $0<\gamma<1$ and $\alpha(t,T)=1-(\frac{t}{T+1})^{\gamma}$ with $\gamma>1$. The discount-rate coefficient $\alpha(t,T)=\rho^{t}$ with $0<\rho<1$ does not satisfy the chronic poverty axiom since it is convex in t.

Rewrite (2.1b) as

$$\begin{aligned}P(\mathbf{x};z) &= \beta(T)\left\{\left[\sum_{t=1}^{T-1}[\alpha(t,T)-\alpha(t+1,T)]\sum_{s=1}^{t}p(x_s;z)\right]+\alpha(T,T)\sum_{s=1}^{T}p(x_s;z)\right\}\\ &\quad +[1-\beta(T)]p(\bar{x};z).\end{aligned} \tag{2.10}$$

and use Abel's partial summation formula one more time (but in reserve order), we have

[6] Recall that the early poverty axiom implies that the weights $\alpha(t,T)$ are non-increasing in t.

Proposition 2.5. For two lifetime consumption profiles $\mathbf{x}$ and $\mathbf{y}$, $P(\mathbf{x}; z) \geq P(\mathbf{y}; z)$ for all $\beta(T) \in [0, 1]$ and all possible values of $\alpha(t, T)$s satisfying the early-poverty axiom and the chronic-poverty axiom if and only if

(1) $l \sum_{t=1}^{T-l+1} p(x_l, z) + \sum_{m=1}^{l-1} (l-m) p(x_{T-l+m+1}, z) \geq l \sum_{t=1}^{T-l+1} p(y_l, z) + \sum_{m=1}^{l-1} (l-m) p(y_{T-l+m+1}, z)$ for $l = 1, 2, ..., T$ and $\min\{\bar{x}, z\} \leq \min\{\bar{y}, z\}$ for a given deprivation function p; or

(2) also for all deprivation functions p such that $p^{'} < 0$, vector

$$\tilde{\mathbf{x}}_l = (\tilde{x}_1, .., \tilde{x}_1, ..., \tilde{x}_{T-l+1}, .., \tilde{x}_{T-l+1}, \tilde{x}_{T-l}, .., \tilde{x}_{T-l}, \tilde{x}_{T-l}, .., \tilde{x}_{T-l}, ..., \tilde{x}_T)$$

is rank dominated by a similarly defined vector $\tilde{\mathbf{y}}_l$ for $l = 1, 2, ..., T$ and $\min\{\bar{x}, z\} \leq \min\{\bar{y}, z\}$; or

(3) also for all deprivation functions p such that $p^{'} < 0$ and $p^{''} > 0$, vector $\tilde{\mathbf{x}}_l$ is generalized Lorenz dominated by vector $\tilde{\mathbf{y}}_l$ for $l = 1, 2, ..., T$ and $\min\{\bar{x}, z\} \leq \min\{\bar{y}, z\}$.

Note that a dominance between $\tilde{\mathbf{x}}_l$ and $\tilde{\mathbf{y}}_l$ does not imply nor is implied by the dominance between $(\tilde{x}_1, .., \tilde{x}_T)$ and $(\tilde{y}_1, .., \tilde{y}_T)$ because none of the vectors is ordered (if they are increasingly ordered then the two types of dominance would be equivalent). An example: $\mathbf{x} = (3, 1)$ and $\mathbf{y} = (2, 4)$ with $z = 5$. Clearly $\mathbf{x}$ is rank ordered by $\mathbf{y}$ but $\tilde{\mathbf{x}}_2 = (3, 3, 1)$ is not rank ordered by $\tilde{\mathbf{y}}_2 = (2, 2, 4)$.

III. The Measurement of a Society's Lifetime Poverty

Consider a society consisting of N individuals who live through the same T periods. These individuals are identical except that they have different lifetime consumption profiles. For each individual i, his consumption profile is $\mathbf{x}^i = (x_1^i, ..., x_t^i, ...x_T^i)'$ and for each period t, the society's consumption distribution is $\mathbf{x}_t = (x_t^1, ..., x_t^i, ...x_t^N)$. The society's lifetime consumption data is represented by a $N \times T$ matrix $X = (\mathbf{x}^1, ..., \mathbf{x}^N) = (\mathbf{x}_1, ..., \mathbf{x}_T)'$. In each period, each individual's poverty status is determined by comparing his consumption level with the poverty line $0 < z < \infty$ which is exogenously given and remains constant throughout the T periods. An individual is poor in period t if and only if his consumption x_t is strictly less than z. An individual's lifetime poverty is $P(\mathbf{x}^i; z)$ and for each period the society's poverty is $P(\mathbf{x}_t; z)$. The society's lifetime poverty level is determined via a function $P(X; z)$ which maps each consumption matrix of the society X into $[0, \infty)$. The average consumption level of individual i over the T periods is $\bar{x}^i$ and the average consumption level of the society at time t is $\bar{x}_t$. As before, we assume that the lifetime poverty measure $P(X; z)$ exhibits properties $\frac{\partial P(X;z)}{\partial x_t^i} \leq 0$ and $\frac{\partial^2 P(X;z)}{\partial (x_t^i)^2} \geq 0$ for all $x_t^i \in [0, \infty)$.

Conceptually, there are two routes to aggregate individual deprivations into a society's lifetime poverty value. The first approach – the person-first approach – is to aggregate each individual's lifetime poverty first – as we have done in the previous section – and then to aggregate across all individuals. Following this approach, the society's lifetime poverty $P(X; z)$ can be written as

$$P(X; z) = \tilde{h}[P(\mathbf{x}^1; z), ..., P(\mathbf{x}^N; z)] \quad (3.1)$$

for some continuous function $\tilde{h}()$ which reflects society's preference about the distribution of lifetime poverty deprivations. The second approach – the society-first approach – is to aggregate society's poverty deprivation in each period and then aggregate these deprivations across all T periods. In this approach, the aggregation of poverty in each period follows the standard approach outlined in the literature of snapshot-poverty measurement and the resulting index satisfies all basic axioms. The second stage of the aggregation follows the steps that we have characterized in the previous section of the paper. In general, we can write the poverty measure derived with the second approach as

$$P(X; z) = k[P(\mathbf{x}_1; z), ..., P(\mathbf{x}_T; z)] \tag{3.2}$$

for some continuous function $k()$ which also reflects society's preference on aggregation over time. Which approach is better? It seems that both are sensible approaches and each can be favorably argued on different grounds, but the two approaches obviously will not always depict the same picture for society's lifetime poverty and its changes.

A natural question is when will both approaches are consistent in depicting the poverty picture of the society over the T periods? Or, in other words, when will the two routes of aggregation lead to the same result? The following path-independence axiom ensures the consistency between the two approaches.

The path-independence axiom. For any consumption matrix X, both the person-first approach and the society-first approach should yield the same level of lifetime poverty for the society.

In aggregating poverty across a population, the axiom of decomposability is commonly used. This axiom allows the overall poverty value to be a weighted average of the poverty levels of subpopulations with the weights being the proportions of population subgroups. Applying such a decomposability requirement to $P(X; z)$, we have for the person-first approach,

$$P(X; z) = h[P(\mathbf{x}^1; z)] + ... + h[P(\mathbf{x}^N; z)] \tag{3.1a}$$

where $h()$ is a continuous function which reflects society's preference on the aggregation process. We require function h to preserve the origin, *i.e.*, $h(0) = 0$ since $P(X; z) = 0$ if $x_t^i \geq z$ for all i and t. This decomposability property is also applicable to the aggregation of society's poverty in each period. For the case of "perfect memory," *i.e.*, $\beta(T) = 1$, we should have

$$P(\mathbf{x}_t; z) = h[p(x_t^1; z)] + ... + h[p(x_t^N; z)]. \tag{3.3}$$

With decomposability, we can characterize a class of path-independent measures of lifetime poverty.

Proposition 3.1. A lifetime poverty measure $P(X; z)$ satisfies both the decomposability axiom and the path-independence axiom if and only if $P(X; z)$ is a positive multiple of

$$P(X;z) = \beta(T)\left\{\sum_{t=1}^{T}\alpha(t,T)\sum_{i=1}^{N}p(x_t^i;z)\right\} + [1-\beta(T)]\sum_{i=1}^{N}p(\bar{x}^i;z). \qquad (3.1b)$$

Proof. For simplicity, consider $T = N = 2$. The path-independence axiom requires that

$$P(X;z) = h[P(\mathbf{x}^1;z), P(\mathbf{x}^2;z)] = k[P(\mathbf{x}_1;z), P(\mathbf{x}_2;z)]. \qquad (3.4)$$

Let $\beta(T) = 1$, then with (2.1b) and the decomposability axiom, (3.4) becomes

$$\begin{aligned} & h[\sum_{t=1}^{2}\alpha(t,T)p(x_t^1;z)] + h[\sum_{t=1}^{2}\alpha(t,T)p(x_t^2;z)] \qquad (3.5) \\ = \; & k\{\sum_{i=1}^{2}h[p(x_1^i;z)], \sum_{i=1}^{2}h[p(x_2^i;z)]\}. \end{aligned}$$

Let $x_2^1 = z$ and $x_2^2 = z$ so that $p(x_2^1;z) = p(x_2^2;z) = 0$, then (3.5) becomes

$$h[\alpha(1,T)p(x_1^1;z)] + h[\alpha(1,T)p(x_1^2;z)] = k\{h[p(x_1^1;z)] + h[p(x_1^2;z)], 0\}.$$

Denote $\omega_1 = \alpha(1,T)p(x_1^1;z)$, $\omega_2 = \alpha(1,T)p(x_1^2;z)$ and $\tilde{k}(\omega_1+\omega_2) = k\{h[\omega_1/a(1,T)] + h[\omega_2/a(1,T], 0\}$, we have

$$h(\omega_1) + h(\omega_2) = \tilde{k}(\omega_1+\omega_2) \qquad (3.6)$$

for all ω_1 and ω_2 in $[0, a(1,T)p(0;z)]$ which is non-empty since both $a(1,T)$ and $p(0;z)$ are strictly positive.

Equation (3.6) is a Pexider equation and its nontrivial solution (Aczél, 1966, p. 142, Theorem 1 and Corollary 1) is:

$$h(\omega_1) = a\omega_1 + b$$

for some constants a and b. But because $h' > 0$ and $h(0) = 0$, b must be 0 and $a > 0$. Substituting $h(\omega_1) = a\omega_1$ into (3.1a) and using (2.1c), (3.1c) is derived. □

When all possible "memory parameter" $\beta(T)$s, the weighting coefficients $\alpha(t,T)$ and/or individual deprivation functions $p(x;z)$ are considered, we have the following results on societal lifetime poverty orderings.

Proposition 3.2. For two lifetime consumption profiles X and Y and a poverty measure $P()$ given in (3.1b), $P(X;z) \geq P(Y;z)$ for all $\beta(T) \in [0,1]$ and all possible values of $\alpha(t,T)$s satisfying the early-poverty axiom and the chronic-poverty axiom if and only if

(1)

$$
\begin{aligned}
& l \sum_{t=1}^{T-l+1} \sum_{i=1}^{N} p(x_l^i; z) + \sum_{m=1}^{l-1} (l-m) \sum_{i=1}^{N} p(x_{T-l+m+1}^i; z) \\
\geq \;& l \sum_{t=1}^{T-l+1} \sum_{i=1}^{N} p(y_l^i; z) + \sum_{m=1}^{l-1} (l-m) \sum_{i=1}^{N} p(y_{T-l+m+1}^i; z)
\end{aligned}
$$

for $l = 1, 2, ..., T$ and $\sum_{i=1}^{N} p(\bar{x}^i; z) \leq \sum_{i=1}^{N} p(\bar{y}^i; z)$ for a given deprivation function p; or

(2) also for all deprivation functions p such that $p' < 0$, vector

$$
\tilde{X}_l = (\tilde{\mathbf{x}}_1, .., \tilde{\mathbf{x}}_1, ..., \tilde{\mathbf{x}}_{T-l+1}, .., \tilde{\mathbf{x}}_{T-l+1}, \tilde{\mathbf{x}}_{T-l}, .., \tilde{\mathbf{x}}_{T-l}, \tilde{\mathbf{x}}_{T-l}, .., \tilde{\mathbf{x}}_{T-l}, ..., \tilde{\mathbf{x}}_T)
$$

is rank dominated by a similarly defined vector $\tilde{Y}_l$ for $l = 1, 2, ..., T$ and $(\min\{\bar{x}^1, z\},..., \min\{\bar{x}^N, z\})$ is rank ordered by $(\min\{\bar{y}^1, z\}, ..., \min\{\bar{y}^N, z\})$; or

(3) also for all deprivation functions p such that $p' < 0$ and $p'' > 0$, vector $\tilde{X}_l$ is generalized Lorenz dominated by vector $\tilde{Y}_l$ for $l = 1, 2, ..., T$ and $(\min\{\bar{x}^1, z\},..., \min\{\bar{x}^N, z\})$ is generalized Lorenz dominated by $(\min\{\bar{y}^1, z\}, ..., \min\{\bar{y}^N, z\})$.

Proof. All results can be derived similarly to those in the previous section and thus are omitted. □

In measuring a society's lifetime poverty, as in any other social welfare measurement over time, income/consumption mobility should be an important factor. It is useful to point out that such a consideration of consumption mobility is contained in the second condition in each case of the proposition. The second condition is the comparison of dominance between two censored mean consumption profiles, *i.e.*, between $(\min\{\bar{x}^1, z\}, ..., \min\{\bar{x}^N, z\})$ and $(\min\{\bar{y}^1, z\}, ..., \min\{\bar{y}^N, z\})$. To see this and to connect with the literature of mobility measurement, it is necessary to introduce the notation of joint distribution among consumptions in different periods. Denote $(x_1, ..., x_t, ...x_T)$ as the consumption profile of a representative individual, $F(x_1, ..., x_t, ...x_T)$ is the joint cdf of one society's consumptions over the T periods and $G(x_1, ..., x_t, ...x_T)$ is the joint cdf of another society's consumption. Suppose $(\min\{\bar{x}^1, z\}, ..., \min\{\bar{x}^N, z\})$ is rank dominated by $(\min\{\bar{y}^1, z\}, ..., \min\{\bar{y}^N, z\})$, then it is equivalent to that $\min\{\bar{y}, z\}$ first-degree stochastic dominates $\min\{\bar{x}, z\}$. It is easy to see that this last dominance is further equivalent to

$$
F(x_1, ..., x_t, ...x_T) \geq G(x_1, ..., x_t, ...x_T) \text{ for all } (x_1, ..., x_t, ...x_T) \text{ such that } \bar{x} < z. \tag{3.7}
$$

For readers who are familiar with the literature of income mobility (*e.g.*, Atkinson and Bourguignon, 1982, Dardanoni, 1993), condition (3.7) is a limited first-degree mobility dominance condition applied to the region of $\{(x_1, ..., x_t, ...x_T)|\bar{x} < z\}$. This mobility factor makes the aggregation of individual lifetime poverty to satisfy a limited version of the familiar "equality-preferring axiom" – the society prefers the individual poverty deprivations to be equally distributed. This is because, all else the same, a more equally distribution of $(\min\{\bar{x}^1, z\}, ..., \min\{\bar{x}^N, z\})$ leads to a lower societal aggregate lifetime poverty.

The first condition (*i.e.*, dominance between $\tilde{X}_l$ and $\tilde{Y}_l$ for $l = 1, 2, ..., T$) does not contain any element of mobility. This is because in (3.1c), individual poverty deprivations are aggregated across society in each time period. As a result, there is no way to trace any individual's consumption over time. The dominance relations between $\tilde{X}_l$ and $\tilde{Y}_l$ are related to the dominances of marginal cdfs such as $F(x_1, \infty, ..., \infty) \geq G(x_1, \infty, ..., \infty)$ for $x_1 < z$. Clearly, this type of dominance among the marginal cdfs does not concern income mobility at all. It is also clear that the dominances of marginal cdfs and (3.7) do not imply each other for $z < \infty$. But as is well known, if $z = \infty$, then condition (3.7) will necessarily imply all the dominances among marginal cdfs such as $F(x_1, \infty, ..., \infty)$ and $F(x_1, x_2, \infty, ..., \infty)$.

But if we follow the person-first approach of aggregation and let the aggregating operator $h()$ in (3.1a) to satisfy $h' > 0$ and $h'' > 0$, *i.e.*, requiring the equity axiom to begin with, consumption mobility will play a role in lifetime poverty comparison even if only snapshot poverty levels are aggregated. To see this, consider $T = 2$ and let $\beta(T) = 1$, then (3.1a) become

$$P(X; z) = \int_0^\infty \int_0^\infty h[\alpha(1,2)p(x_1; z) + \alpha(2,2)p(x_2; z)]dF(x_1, x_2). \tag{3.1c}$$

Integrating $P(X; z)$ by parts twice (similar to what is done in Atkinson and Bourguignon (1982)) and using the condition $p(x_t; z) = 0$ for $x_t \geq z$, we have

$$\begin{aligned} P(X; z) &= -\int_0^z \{h'\alpha(1,2)p'(x_1; z)\}F(x_1, \infty)dx_1 - \int_0^z h'\{\alpha(2,2)p'(x_2; z)\}F(\infty, x_2)dx_2 \\ &\quad + \int_0^{2z} \int_0^{2z-x_1} \{h''\alpha(1,2)\alpha(2,2)p'(x_1; z)p'(x_2; z)\}F(x_1, x_2)dx_2dx_1. \end{aligned}$$

Since $h' > 0$ and $h'' > 0$, then for all possible values of $\alpha(1,2)$ and $\alpha(2,2)$, the necessary and sufficient condition for lifetime poverty to be higher in distribution F than in G for all poverty measures (3.1c) with $p' < 0$ would be

$$\begin{aligned} F(x_1, \infty) &\geq G(x_1, \infty) \text{ for all } x_1 < z;\ F(\infty, x_2) \geq G(\infty, x_2) \text{ for all } x_2 < z \\ \text{and } F(x_1, x_2) &\geq G(x_1, x_2) \text{ for all } x_1 \text{ and } x_2 \text{ such that } x_1 + x_2 < 2z. \end{aligned}$$

If $\alpha(1,2) \geq \alpha(2,2)$, then for all possible such values of $\alpha(1,2)$ and $\alpha(2,2)$, the necessary and sufficient condition for lifetime poverty to be higher in distribution F than in G for all poverty measures (3.1c) with $p' < 0$ would be

$$\begin{aligned} F(x_1, \infty) &\geq G(x_1, \infty) \text{ for all } x_1 < z; \\ F(x_1, \infty) + F(\infty, x_1) &\geq G(x_1, \infty) + G(\infty, x_1) \text{ for all } x_1 < z; \\ \text{and } F(x_1, x_2) &\geq G(x_1, x_2) \text{ for all } x_1 \text{ and } x_2 \text{ such that } x_1 + x_2 < 2z. \end{aligned}$$

In both sets of conditions, consumption mobility is reflected in the requirement $F(x_1, x_2) \geq G(x_1, x_2)$ over $x_1 + x_2 < 2z$. Note that these dominance conditions can be generalized to more than two periods and for all poverty deprivation functions satisfying $p' < 0$ and $p'' > 0$, but additional conditions of higher orders (all the way up to the $2T$th order) must be specified on the h function.

IV. Some Remarks and Conclusion

In all of our discussions and derivations except Propositions 2.2 and 2.3, the early-poverty axiom has been assumed. Recall that this axiom assumes that the poverty weighting function $\alpha(t, T)$ is a nonincreasing function of time t. Although we believe this axiom is appealing from the point of view that earlier-life poverty may impact latter-life standards of living but not the other way around, it doe not mean that the axiom will be universally accepted. In fact, one may argue that poverty may matter more when the time is getting closer to T. Or one may even suggest that poverty matters more during both ends of life. In the former case, the weighting function will exhibit

$$\alpha(1, T) \leq \alpha(2, T) \leq ... \leq \alpha(T, T), \tag{4.1}$$

and for the latter case it is

$$\alpha(1, T) \geq \alpha(2, T) \geq ... \geq a(t^* - 1, T) \geq a(t^*, T) \leq a(t^* + 1, T) \leq ... \leq \alpha(T, T) \tag{4.2}$$

for some given period t^* in the middle.

For both alternative stipulations of $\alpha(t, T)$, the corresponding dominance conditions can be derived. For all poverty measures satisfying (4.1) and two lifetime consumption profiles $\mathbf{x}$ and $\mathbf{y}$, $P(\mathbf{x}; z) \geq P(\mathbf{y}; z)$ if and only if

$$\sum_{t=l}^{T} p(x_t, z) \geq \sum_{t=l}^{T} p(y_t, z) \text{ for } l = 1, 2, ..., T \text{ and } \min\{\bar{x}, z\} \leq \min\{\bar{y}, z\};$$

the two other conditions for all possible poverty deprivation functions $p()$ can be derived similarly to those of Proposition 2.4. For condition (4.2), a sufficient condition would be

$$\begin{aligned} \sum_{t=1}^{l} p(x_t, z) &\geq \sum_{t=1}^{l} p(y_t, z) \text{ for } l = 1, 2, ..., t^*, \\ \sum_{t=l}^{T} p(x_t, z) &\geq \sum_{t=l}^{T} p(y_t, z) \text{ for } l = t^*, 2, ..., T \text{ and } \min\{\bar{x}, z\} \leq \min\{\bar{y}, z\}. \end{aligned}$$

There are other important issues on societal aggregation of individual poverty deprivations that we leave for further study. These include how to compare individuals who live over different numbers of periods (e.g., overlapping generations), for people who live a different number of years, etc. Also, data on consumption is not always available while data on incomes is. This poses further challenges for implementation of measuring lifetime poverty for any approach, including ours.

Our formulation of lifetime poverty measurement provides an axiomatic framework applied to poverty experiences over multiple periods of an individual's life. Besides recognizing that an individual's lifetime poverty should be reflected both by poverty experiences of each period (i.e., "snapshot" poverty) and the poverty level of "permanent" lifetime consumption, we also introduce axioms to reflect the sensitivity of how any poverty spells are distributed over a person's lifetime. In particular, we investigate the implications of adopting an "early poverty axiom" and a "chronic poverty axiom". The early poverty axiom reflects the popular notion that poverty early in life is more critical than poverty later in life due to a carry-over impact this can have on an individual's well-being later in life. The chronic poverty axiom reflects the idea that, for example, two successive spells of poverty of a given intensity are more harmful to an individual's well-being than two spells separated by a period (or periods) of non-poverty.

In conjunction with other axioms that we believe are compelling, such as a strong monotonicity axiom and an independence axiom, we develop a number of results characterizing how one can measure lifetime poverty through the use of two components; one being a weighted average of all snapshot poverty levels and another being the poverty of average lifetime consumption. The use of discounting weights to aggregate over snapshot poverty experiences is consistent with the early poverty axiom but is not necessarily implied by a concern with early poverty. Moreover, discounting weights are not consistent once one introduces our chronic poverty axiom. We provide examples of weighting functions that are consistent with both axioms and demonstrate some general properties that such weighting functions must satisfy. We also develop necessary dominance conditions, relating to first and second degree stochastic dominance, that satisfy various combinations of our axioms. We believe these results will be useful in focussing attention on how to properly capture the intent of measurement of lifetime poverty experiences as well as providing insight on how to compare poverty alleviation programs that may have different impacts on atemporal and temporal aspects of poverty.

Finally, we consider a path-independence requirement to make the two alternative approaches of aggregation consistent.

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